Perturbing Doubly Stochastic Matrices

Philip A. Knight\textsuperscript{1} Daniel Ruiz\textsuperscript{2}

\textsuperscript{1}Department of Mathematics and Statistics
University of Strathclyde

\textsuperscript{2}ENSEEIHT

May 2019
Scaling to doubly stochastic form

- $A \geq 0$.
- Seek doubly stochastic matrix $P = DAF$ where $D$ and $F$ are diagonal matrices.
- Unique scaling (up to scalar multiple) if $A$ is fully indecomposable (FI).
- Multiple scalings exist if $A$ has total support.
- Use scaling as a tool to discover hidden block structure.
Suppose that $S \in \mathbb{R}^{n \times n}$ is FI, and doubly stochastic.

- Principal singular value of $S$ is 1, with multiplicity 1.
- Associated singular vector is equal to a multiple of $e$.
- Suppose $S$ is a permutation of a block diagonal structure with $k$ blocks.
- Singular value 1 has multiplicity $k$.
- Partition $S$ according to the blocks and partitioned components of singular vectors are constant.
Simple Block Detection Scheme

- Compute a principal singular vector $\mathbf{x}$.
  $$\mathbf{x} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_k \mathbf{v}_k.$$
- Characterise the partitions exactly using the set $\{a_1, \ldots, a_k\}$.
- If the matrix has a structure that is close to block diagonal, then we hope the leading singular vectors have a similar structure.
- Look for steps in the computed vectors to reveal an underlying near block structure.
- Akin to Fiedler vectors.
Let $P = RAC$.

Consider a small perturbation to $A$, namely $\tilde{A} = A + \epsilon H$.

How close is the scaling of $\tilde{A}$ to $P$?

We’d like to assume that $A$ is block diagonal matrix.

Start by assuming that $A$ is FI.
Theorem

- $A, \tilde{A}, R, C, H$ and $P$ as described above.
- $\tilde{P} = P + \epsilon Q$ where $Q = RHC$.
- There exist vectors $f$ and $g$ such that $\|f\|, \|g\| = O(\epsilon)$.
- If $F = D(e + f)$, $G = D(e + g)$, then
  \[
  \begin{bmatrix}
  F\tilde{P}Ge \\
  G\tilde{P}^TFe
  \end{bmatrix} = e + m.
  \]
- $\|m\| = O(\epsilon^2)$. 
Proof

- Key is to find \( f \) and \( g \) so that

\[
\begin{bmatrix}
P_{sym}
\end{bmatrix}
\begin{bmatrix}
f \\
g
\end{bmatrix}
= 
\begin{bmatrix}
I & P \\
P^T & I
\end{bmatrix}
\begin{bmatrix}
f \\
g
\end{bmatrix}
= -\epsilon
\begin{bmatrix}
Qe \\
Q^Te
\end{bmatrix}.
\]

- Not automatic as we know that \( P_{sym} \) is singular.

- Since \( A \) is FI, kernel is 1D with basis \( \begin{bmatrix} e \\ -e \end{bmatrix} \).

\[
\begin{bmatrix}
e \\
-e
\end{bmatrix}^T
\begin{bmatrix}
Qe \\
Q^Te
\end{bmatrix}
= e^TQe - e^TQ^Te = 0.
\]

- Choice of \( f \) and \( g \) is motivated by the result of applying Newton method to \( \tilde{P} \) with initial vector \( e \).
Is $P$ close to $\tilde{P}$?

- Use theorem to give precise conditions under which $A$ and $\tilde{A}$ have nearby doubly stochastic scalings.
- $\|F\tilde{P}G - P\|$ is small when $\|f\|$ and $\|g\|$ are small.
- This is true if $\|Q\|$ is constrained and the smallest nonzero eigenvalue of $P_{sym}$ is bounded away from zero.
- These are reasonable expectations in the context of uncovering block structure.
Extending The Result

- We need to extend our analysis to cover the case when $A$ is not FI but has total support.
- This occurs observe when $A$ has perfect block structure.
- Unit singular value has multiplicity equal to number of blocks.
- Extra analysis needed to show system involving $P_{sym}$ is consistent.
Two Blocks

\[ P = \begin{bmatrix} P_1 & O \\ O & P_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \]

Kernel spanned by \( e_x = \begin{bmatrix} e_1 \\ 0 \\ -e_1 \\ 0 \end{bmatrix}, \quad e_y = \begin{bmatrix} 0 \\ e_2 \\ 0 \\ -e_2 \end{bmatrix}. \)

\[ \begin{bmatrix} e_x^T \\ e_y^T \end{bmatrix} \begin{bmatrix} Qe \\ QT e \end{bmatrix} = \begin{bmatrix} e_1^T Q_{12} e_2 - e_1^T Q_{21} e_2 \\ e_2^T Q_{21} e_1 - e_2^T Q_{12} e_1 \end{bmatrix}. \]

Diagonal factors for \( P_1 \) and \( P_2 \) are completely decoupled.

Gives a degree of freedom which allows us to make inner product zero.

If there are more than two blocks then we can establish consistency recursively a block at a time.
If we make a small perturbation to a matrix with block structure then it is reasonable to assume that the associated doubly stochastic matrices are also close together.

We would like to draw corresponding conclusions about the corresponding singular vectors but so far we can only provide empirical evidence of this.