Wilkinson’s bus: Weak condition numbers, with applications

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Joint with Martin Lotz (Warwick)

J.H.Wilkinson’s 100th birthday, Manchester

May 30th, 2019
Wilkinson’s bus

Wilkinson on the rarity of worst-case scenarios, speaking of Gaussian Elimination:
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Anyone that unlucky has already been run over by a bus.

(I learnt the quote by reading N. Trefethen, The Smart Money’s on Numerical Analysts, SIAM News 45(9), 2012)
Classical condition number

\[ f : \mathcal{V} \rightarrow \mathcal{W} \]

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- Morally \( \kappa \) should predict (in a worst case sense) the accuracy of computations in a fixed finite precision arithmetic setting
A generalized eigenproblem

\[ L(x) = \begin{bmatrix} -1 & 1 & 4 & 2 \\ -2 & 3 & 12 & 6 \\ 1 & 3 & 11 & 6 \\ 2 & 2 & 7 & 4 \end{bmatrix} x + \begin{bmatrix} 2 & -1 & -5 & -1 \\ 6 & -2 & -11 & -2 \\ 5 & 0 & -2 & 0 \\ 3 & 1 & 3 & 1 \end{bmatrix} \]

MATLAB R2016a's solution:

```matlab
» eig(L0,-L1)
ans =
   -138.1824366539536
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Observations

QZ is not a structured algorithm such as e.g. staircase! As a consequence it is blind to the fact that $L(x)$ has only one eigenvalue, because almost all $4 \times 4$ pencils have four. Hence it is expected that out of the four computed eigenvalues three are just noise. This cannot be avoided (although one can check reliability a posteriori... more to come).

The problem is ill posed (=discontinuous): plenty of pencils arbitrarily close to $L(x)$ whose eigenvalues are all nowhere near 1: definitely $\kappa = \infty$, and arguably of the worst kind. Yet the eigenvalue 1, in spite of its infinite condition, is computed with full accuracy! How come?
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For our $L(x)$: $\kappa_s = \infty$. Average case analysis does not explain why QZ computes 1 so well.
First ingredient to go beyond

Directional sensitivity:

\[ \sigma_E = \lim_{\epsilon \to 0} \frac{\| f(D + \epsilon E) - f(D) \|}{\epsilon \| E \|} \]

- Ratio of forward and backward errors for a particular direction of the backward error
- If \( f \) is differentiable, \( \| E \|^{-1} \) times the norm of the Gateaux derivative
Weak condition number

We want to do some probability, so we fix a probability space \((\Omega, \Sigma, \mathbb{P})\) and a random variable \(E : \Omega \to V\).

In practice, "\(\kappa_w(\delta)\) is bounded above by \(b(\delta)\)" means: with probability \(1 - \delta\), the forward error is bounded by \(b(\delta)\) times the backward error.

The probability distribution can be seen as a parameter of this definition, and can be made more concrete according to context. \(\delta\) can be seen as a very concrete parameter (confidence level) to be input by the user (engineer, scientist, mathematician).

We do not wish to model rounding errors probabilistically, but to argue that the set of bad perturbations may be so small that algorithms would need a good reason to stumble on it.
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Given \(0 \leq \delta < 1\), the \(\delta\)-weak worst-case condition is

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Theorem (Lotz, VN)

With respect to uniformly distributed real perturbations on the unit sphere, the weak condition of the eigenvalue 1 of $L(x)$ is bounded by

$$\kappa_w(\delta) \leq \max\{12.16, \frac{2.149}{\delta}\}.$$
Back to $L(x)$

**Theorem (Lotz, VN)**

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(In the paper: way more general results for any simple eigenvalue of any matrix polynomial, singular or regular, for both real and complex perturbations.)
Is this practical?

Yes. The general result depend on a parameter $\gamma_P$ that generalizes Tisseur’s formula for eigenvalue condition of regular matrix polynomials, but is trickier to compute exactly for singular matrix polynomials because eigenvectors are only defined in certain quotient spaces (as opposed to traditional vector spaces).

However $\gamma_P$ can be cheaply estimated in practice. All we need is the computed eigentriple from QZ. No more expensive than computing traditional condition numbers for regular polynomial eigenvalue problems, and this can tell reliable eigenvalues from rubbish (remember the 3 spurious computed eigenvalues).

For more details: arxiv.org/pdf/1905.05466.pdf
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