Three Precision GMRES-Based Iterative Refinement for Least Squares Problems

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Joint work with

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IR for least squares

Given $A \in \mathbb{R}^{m \times n}$, $m \geq n$ and full rank, and b solve $\min_x \|b - Ax\|_2$

- $r = b - A\widehat{x}$.
- Solve $\min_d \|b - Ad\|_2$.
- Update $y = \widehat{x} + d$.

Analyzed by Golub and Wilkinson (1965).

Conclusion

Works well only for nearly consistent systems.

Björk's idea

Augmented system

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$\widetilde{A}\widetilde{x} = \widetilde{b}$$

- Invoke iterative refinement of a linear system.
- Linear system can be solved using QR factors.

Question

How to invoke GMRES-IR, using low precision QR factorization?

Recall

$$u_f \geq u \geq u_r$$

Given $\widetilde{A}$ and $\widetilde{b}$ in precision u.

Solve $\widetilde{A}\widetilde{x}_0 = \widetilde{b}$ in precision u_f.u_f

- $r = \widetilde{b} - \widetilde{A}\widetilde{x}_0$, in u_r.
- Solve $\widetilde{A}d = r$, at precision u.
- Update $x_1 = \mathrm{fl}(\widetilde{x}_0 + d)$ in precision u.

Variants

- Type 1 – Solve the augmented system using u_f precision QR factors.
- Type 2 – GMRES with an appropriate preconditioner.

Convergence Conditions

Properties of correction equation solver to ensure convergence

- Solutions of some relative accuracy.
- Small normwise b'err.
- Small componentwise b'err.

Householder QR is assumed

- Type 1 –
  - f'err $\leq c_{m,n} u_f \kappa_\infty(A)$
  - $u_f =$half, $\kappa_\infty(A) \leq 2 \times 10^3$.

Preconditioner

Scaling –

$$\widetilde{A}_\alpha = \begin{bmatrix} \alpha I & A \\ A^T & 0 \end{bmatrix}, \quad \kappa_2(\widetilde{A}_\alpha) = \mathcal{O}(\kappa_2(A)).$$

if $\alpha = 2^{-1/2}\sigma_{\min}(A)$.

Preconditioner –

$$M = \begin{bmatrix} \alpha I & QR \\ R^T Q^T & 0 \end{bmatrix}, \quad M^{-1} \begin{bmatrix} \frac{1}{\alpha}(I - QQ^T) & QR^{-T} \\ R^{-1}Q^T & -\alpha R^{-1}R^{-T} \end{bmatrix}.$$

Condition number

$$\kappa_\infty(M^{-1}\widetilde{A}) \lesssim (1 + (\cdot)u_f\kappa_\infty(A))^2$$

Type-2

Condition number

$$\text{f'err} \leq u f(m+n) \kappa_\infty(M^{-1}\widetilde{A})$$

$$\kappa_\infty(A) < u^{-1/2} u_f^{-1}$$

u_f	u	u_r	$\kappa_\infty(A)$	Backward error Norm.	Backward error Comp.	$\kappa_\infty(A)$	Forward error
half	half	single	$2 \cdot 10^3$	half	half	10^5	half
half	single	single	10^3	single	single	10^4	$\text{cond}(\widetilde{A}, y) \cdot 10^{-7}$
half	single	double	10^7	single	single	10^7	single
half	double	double	10^6	double	double	10^7	$\text{cond}(\widetilde{A}, y) \cdot 10^{-16}$
half	double	quad	10^{16}	double	double	10^{11}	double

Numerical Experiments

- 100×10, mode 3, `randsvd` matrices.
- SuiteSparse: $20 \leq m \leq 2000$, $n \leq 400$, $n < m$.
- (half,double,quad)
- Advanpix for quad precision.
- `chop` for fp16.
- IR terminated for f'err $\leq nu$.
- α is computed using low precision R factor.

f'err is for augmented system.

randsvd

$\kappa_2(A)$	LSIR	GMRES-LSIR
1.00e+02	13	2 (16)
1.00e+04	–	2 (24)
1.00e+07	–	2 (70)
1.00e+09	–	4 (212)
1.00e+10	–	5 (375)
1.00e+11	–	13 (992)

Suite Sparse

Index	LSIR	GMRES-LSIR
1	7	1 (6)
2	9	2 (14)
3	5	1 (5)
4	11	2 (16)
5	5	1 (5)
6	–	4 (1616)
7	5	1 (5)
8	5	1 (5)
9	–	2 (89)
10	–	2 (28)
11	–	4 (5428)

Remarks

- GMRES-LSIR enhances the class of problems that can be solved using low precision QR factorization.
- Underflow and overflow can be addressed using one sided diagonal scaling.
- Similar results for (half,single,double), and (single,double,quad).
- *Other saddle preconditioners can be tried preconditioners.
- *MINRES as the iterative solver for corrections equation.

Conclusion

- GMRES-LSIR is suitable for solving ill conditioned problems.
- Most of the work is done in low precision matrix factorization.
- Two sided preconditioning works well in practice.

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Carson, Higham, **P**,*'Three Precision GMRES-Based Iterative Refinement for Least Squares Problems'*, To be Submitted.

Thank you!
Questions?