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What Is the Gerstenhaber Problem?

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Matrix Space

- Vector space $\mathbb{R}^{n\times n}$ of real matrices has dimension n^2, with a basis $E_{ij} = e_i e_j^T$.
- Cayley–Hamilton theorem says that $p(A) = 0$ where $p(t) = \det(\lambda I - A)$ is the characteristic polynomial of A.
- A^n can be expressed as a linear combination of I, A, ..., A^{n-1}, so the powers of A span a vector space of dimension at most n.

Gerstenhaber's 1961 Result

Theorem

If A and B are two commuting $n \times n$ matrices then the matrices $A^i B^j$, $0 \leq i, j \leq n$, generate a vector space of dimension at most n.

Gerstenhaber Problem

For **three** commuting $n \times n$ matrices A, B, and C can the vector space

$$S_n = \{ A^i B^j C^k : 0 \le i, j, k \le n-1 \}$$

have dimension greater than n?

No for Four

Take 4×4 matrices

$$A = e_1 e_3^T, \quad B = e_1 e_4^T, \quad C = e_2 e_3^T, \quad D = e_2 e_4^T$$

all possible products are zero, so the matrices commute pairwise. Yet $I = A^0, A, B, C, D$ are clearly **five** linearly independent matrices.

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Is Gerstenhaber like the FLT:
"true for one and two, fails for three or more"?

Some Known Facts

- The result holds for all $n \leq 11$.
- Failure for one value of n implies failure for all larger n.
- By a 1905 result of Schur, the dimension of S_n is at most $1 + \lfloor n^2/4 \rfloor$.

Holbrook & O'Meara (2020) state that they

> "firmly believe the GP will turn out to have a negative answer".

Computational Search for Counterexample

For some $n \geq 12$, choose three commuting $n \times n$ matrices A, B, and C, select $m \geq n$ monomials

$$X_i = A^{i_p} B^{j_p} C^{k_p}, \quad 1 \leq p \leq m,$$

form the matrix

$$Y = [\mathrm{vec}(X_1), \mathrm{vec}(X_2), \ldots, \mathrm{vec}(X_m)],$$

then compute rank(Y), which is a lower bound on $\dim(S_n)$, and check whether it exceeds n.

Challenges

- How do we choose A, B, and C?
- How do we choose the powers?
- How do we avoid overflow and underflow and compute a reliable rank, given that we might be dealing with large powers of large matrices?

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Holbrook and O'Meara mostly tried $m \leq 50$ but feel that

> "the Loch Ness monster probably lives in deeper water, closer to 100×100."

Searching for a Eureka

Holbrook and O'Meara (2020), call a case with $\dim(S_n) > n$ a

Eureka.

They note that:

- A, B, and C can be assumed to be nilpotent (hence defective).
- Since commuting matrices are simultaneously unitarily triangularizable, A, B, and C can be assumed to be strictly upper triangular.
- A can be assumed to be in **Weyr canonical form**.

Weyr Canonical Form

- A dual of the Jordan canonical form.
- Jordan matrix replaced by a Weyr matrix, which is a direct sum of basic Weyr matrices.

$$W(\lambda) = \left[\begin{array}{cc|cc|c} \lambda & 0 & 1 & 0 & \\ 0 & \lambda & 0 & 1 & \\ \hline & & \lambda & 0 & 1 \\ & & 0 & \lambda & 0 \\ \hline & & & 0 & \lambda \end{array}\right], \quad J(\lambda) = \left[\begin{array}{ccc|cc} \lambda & 1 & 0 & & \\ 0 & \lambda & 1 & & \\ 0 & 0 & \lambda & & \\ \hline & & & \lambda & 1 \\ & & & 0 & \lambda \end{array}\right].$$

$J(\lambda) = P^T W(\lambda) P$ for some permutation matrix P.

Form W from J via a dot diagram (Young diagram):

	j_1	j_2
w_1	●	●
w_2	●	●
w_3	●	

Why Weyr?

- Any matrix that commutes with a **Jordan matrix** is a **Toeplitz matrix**.
- Any matrix that commutes with a **Weyr matrix** is **block upper triangular**.
- From A in Weyr, commuting matrices B and C can be built up in a systematic way.

Codes

- *Suffices to compute modulo a prime p* (**O'Meara**, 2020) so the computations can be done in exact arithmetic.
- **Holbrook & O'Meara** (2020) have MATLAB codes available on request.
- 41 codes, nicely commented but not optimized.
- Most of time is spent in computing GCDs:

```
while a ~= 0
    for i = 40:-1:0
        while  ((10)^i)*b < a
            a = a - ((10)^i)*b;  % subtract biggest
        end
    end
```

References I

R. M. Corless and S. E. Thornton.
The Weyr canonical form.
https://s3.amazonaws.com/stevenethornton.github/WeyrForm.pdf, 2016.

J. Holbrook and K. C. O'Meara.
A computing strategy and programs to resolve the Gerstenhaber problem for commuting triples of matrices.
ArXiv:2006.08588,, June 2020.

K. C. O'Meara.
The Gerstenhaber problem for commuting triples of matrices is "decidable".
Comm. Algebra, 48(2):453–466, 2020.