MIXED PRECISION LU FACTORIZATION ON GPU TENSOR CORES

Reducing Data Movement and Memory Footprint

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SIAM CSE'21

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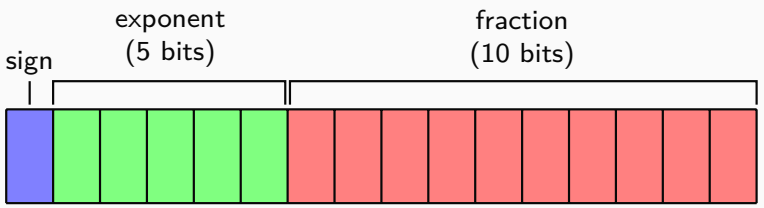
Today's floating-point precision arithmetics

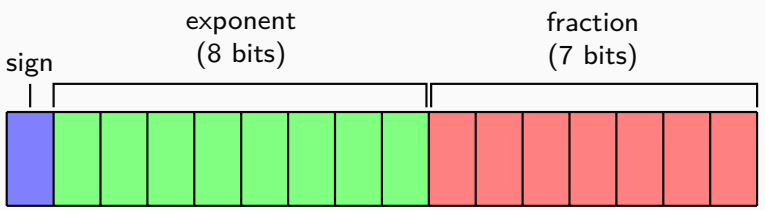
Type		Bits	Range	$u = 2^{-t}$
`fp128`	`quad`	128	$10^{\pm 4932}$	$2^{-113} \approx 1 \times 10^{-34}$
`fp64`	`double`	64	$10^{\pm 308}$	$2^{-53} \approx 1 \times 10^{-16}$
`fp32`	`single`	32	$10^{\pm 38}$	$2^{-24} \approx 6 \times 10^{-8}$
`fp16`	`half`	16	$10^{\pm 5}$	$2^{-11} \approx 5 \times 10^{-4}$
`bfloat16`	`half`	16	$10^{\pm 38}$	$2^{-8} \approx 4 \times 10^{-3}$

Half precision increasingly **supported by hardware**:

- Present: **NVIDIA** Pascal, Volta & Ampere GPUs, **AMD** Radeon Instinct MI25 GPU, **Google** TPU, **ARM** NEON
- Near future: Fujitsu A64FX ARM, **IBM** AI chips, **Intel** Xeon Cooper Lake and Intel Nervana Neural Network



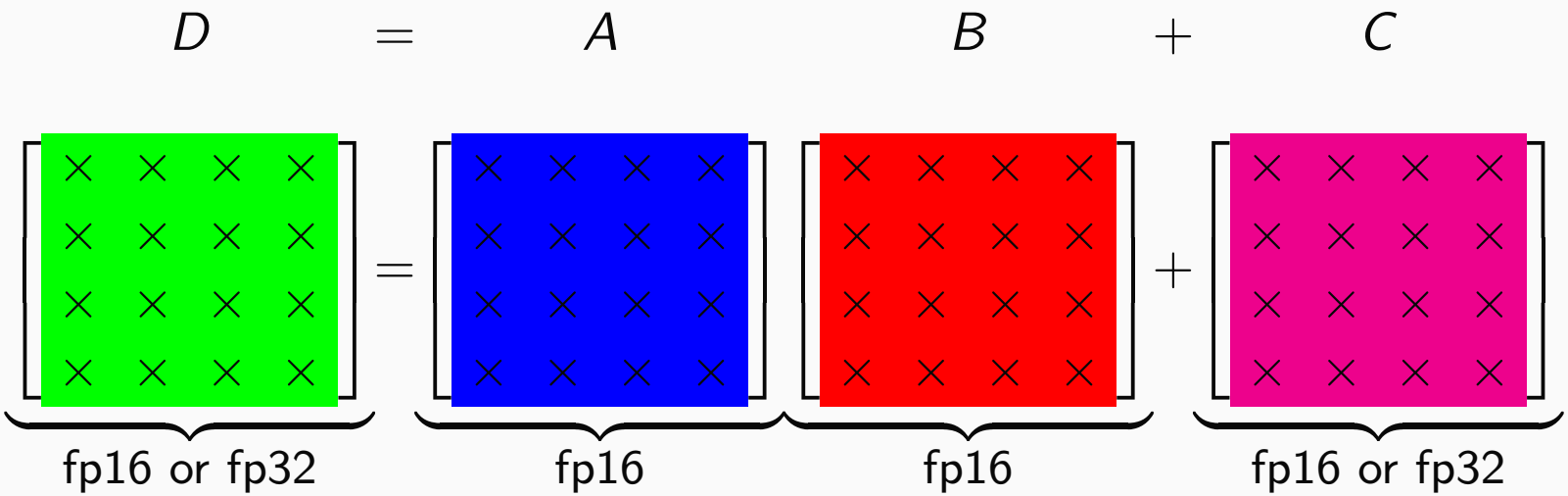
fp16



bfloat16

NVIDIA GPU tensor cores units

Block FMA: 4×4 matrix multiplication **in 1 clock cycle**:



Performance peak: 125 TFlop/s (8× speedup vs fp32)

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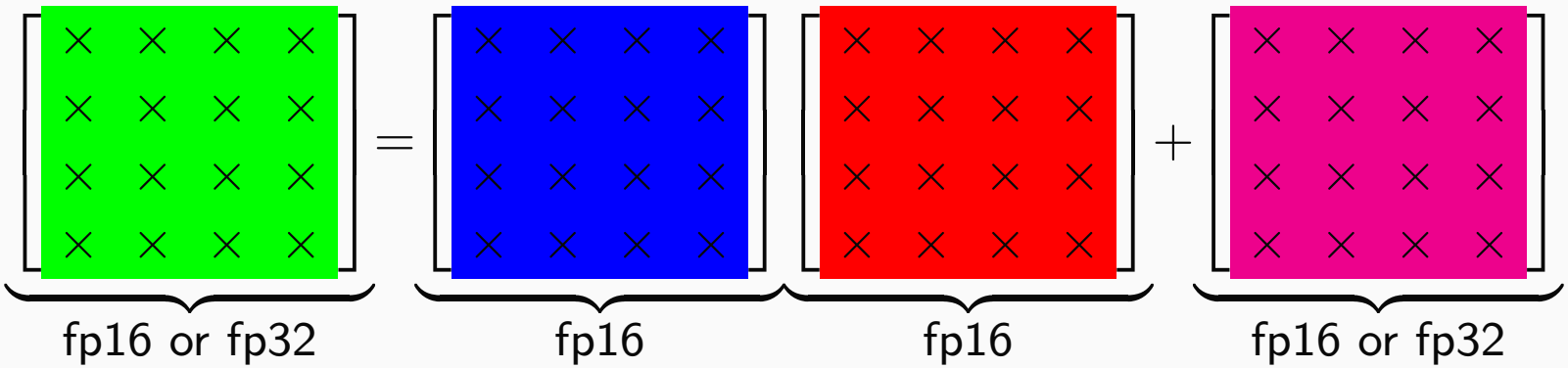
$$D = A \quad B + C$$



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For the matrix product $C = AB$ where $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ using tensor cores, the computed $\widehat{C}$ satisfies (Blanchard et. al[1]):

$$|\widehat{C} - C| \lesssim c_n |A||B|, \quad c_n = \left\{ \right.$$

[1] P. Blanchard, N. J. Higham, F. Lopez, T. Mary and S. Pranesh. *Mixed precision block fused multiply-add: Error analysis and application to GPU tensor cores* (SIAM SISC 2020)

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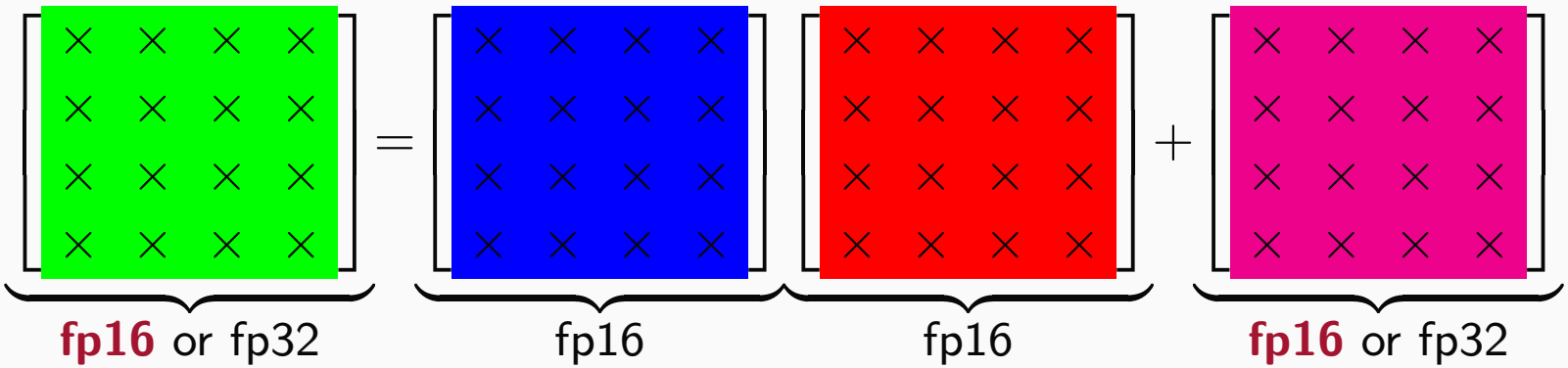

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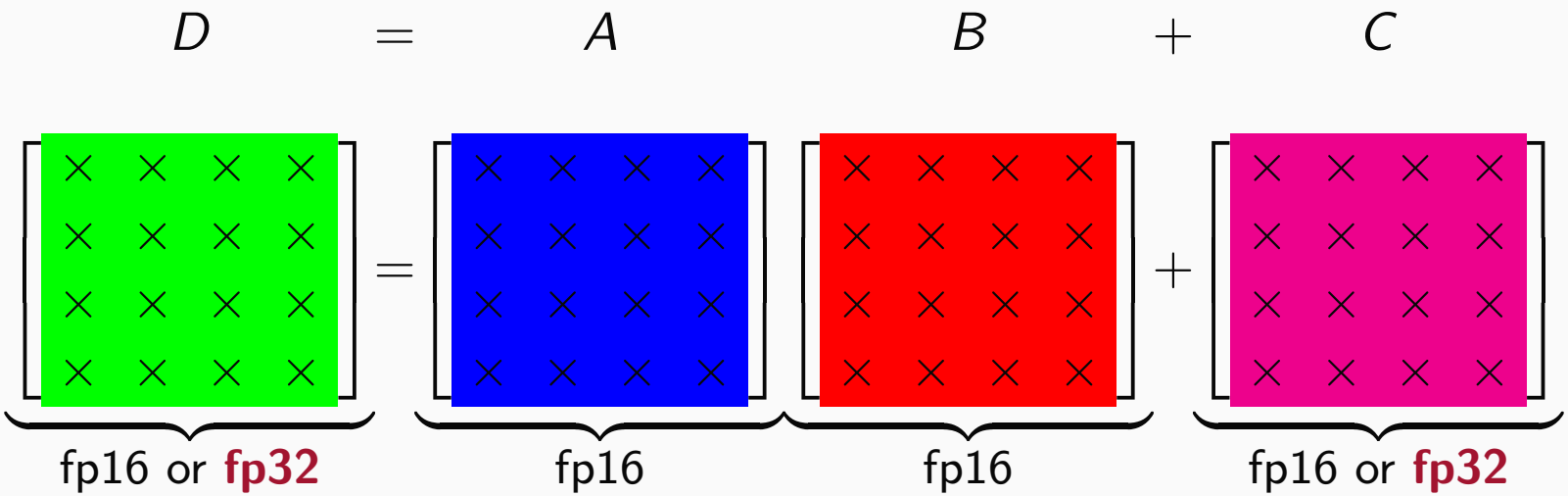

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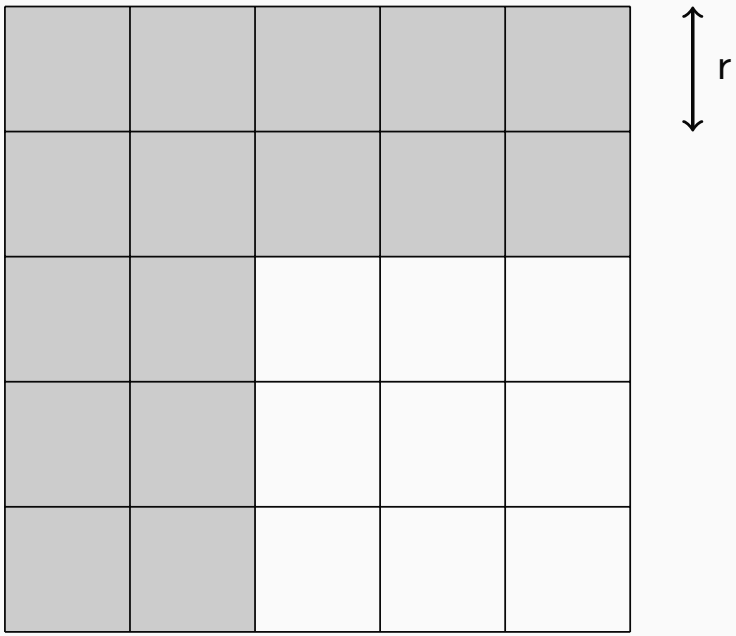
Partitioned LU factorization: right-looking variant

This algorithm computes $A = LU$ where $A \in \mathbb{R}^{n \times n}$, stored in precision $u = u_{16}$ or $u = u_{32}$, is partitioned into $r \times r$ blocks $\Rightarrow$ matrix–matrix (Level-3 BLAS) operations.



for $k = 1 : q$ **do**
  Factorize $L_{kk} U_{kk} = A_{kk}$.
  for $i = k + 1 : q$ **do**
    Solve $L_{ik} U_{kk} = A_{ik}$ for L_{ik}.
    Solve $L_{kk} U_{ki} = A_{ki}$ for U_{ki}.
  end for
  for $i = k + 1 : q$ **do**
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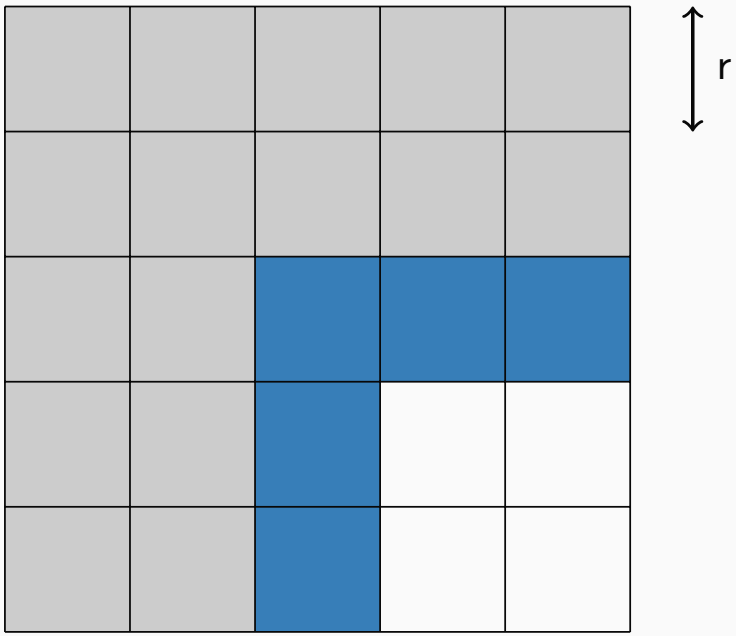

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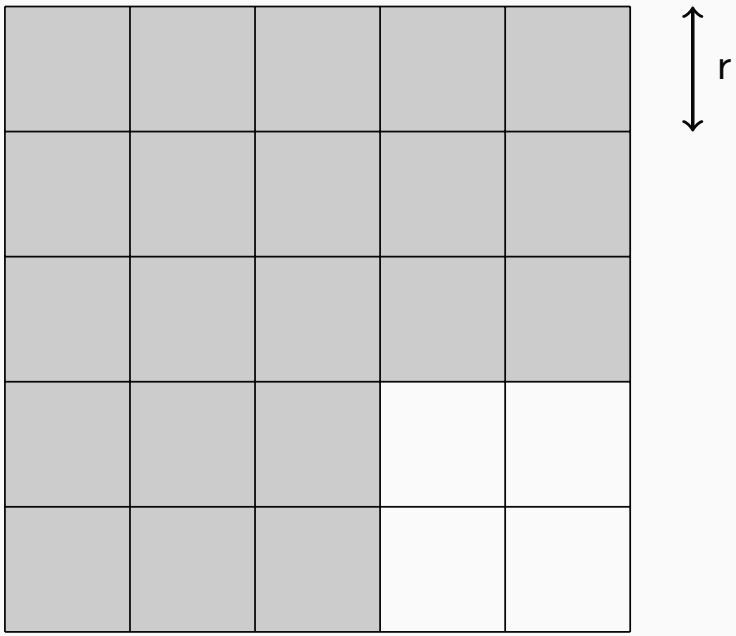

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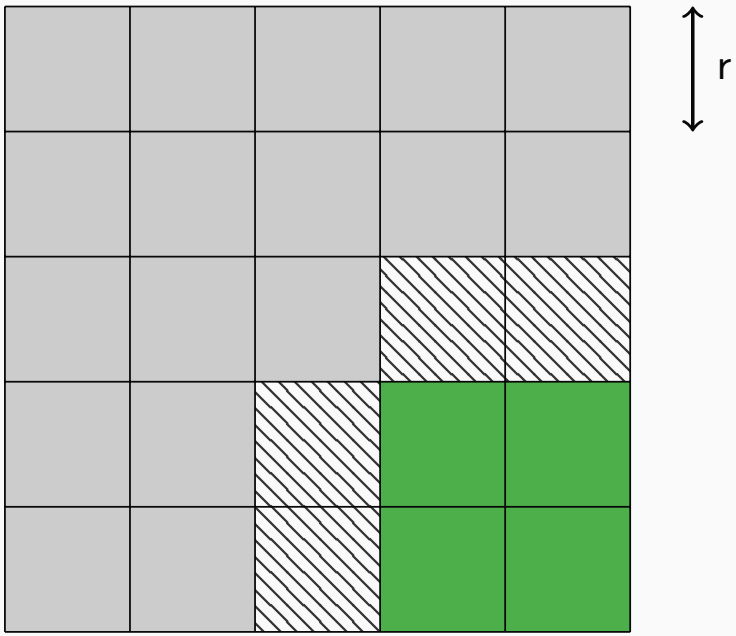

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State-of-the-art mixed precision LU factorization

Mixed precision LU factorization algorithm proposed in Haidar et al [2]:

for $k = 1 : n/r$ **do**
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  for $i = k+1 : n/r$ **do**
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      Convert to fp16: $\widetilde{L}_{ik} \leftarrow \mathrm{fl}_{16}(L_{ik})$ and $\widetilde{U}_{ki} \leftarrow \mathrm{fl}_{16}(U_{ki})$.
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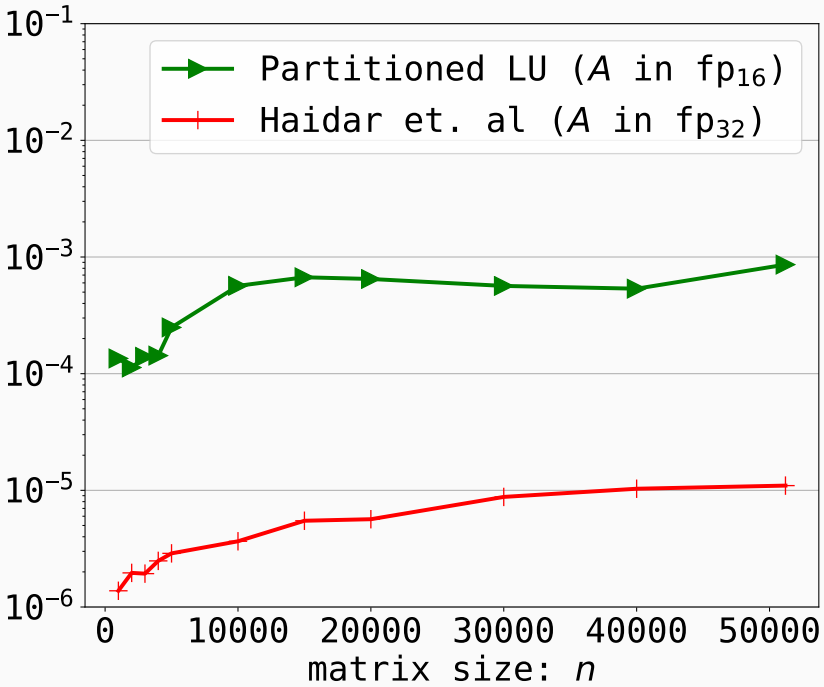
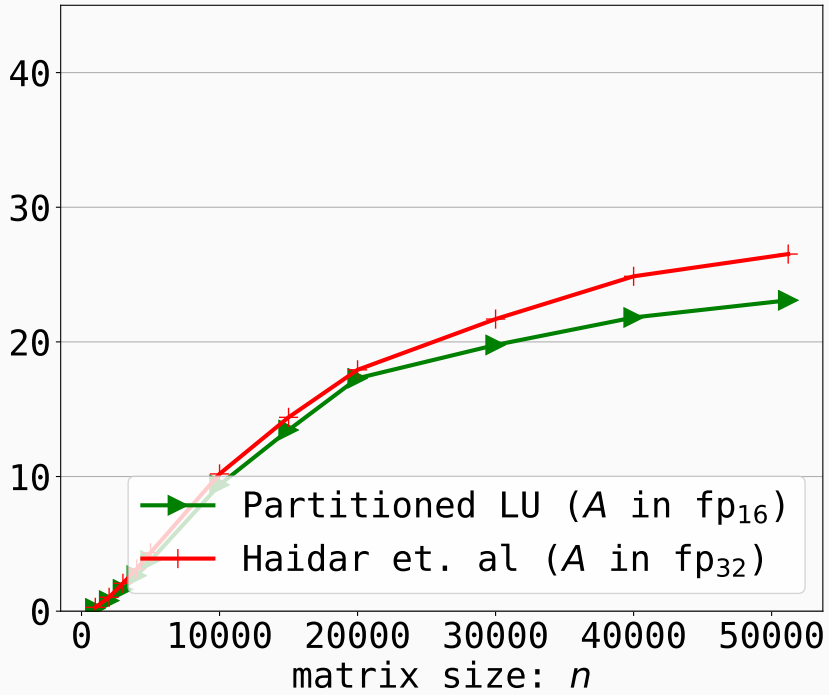
Backward error bounds from Blanchard et. al

Partitioned LU $u = u_{16}$	Haidar et. al $u = u_{32}$
nu_{16}	$2u_{16} + nu_{32}$



Backward error



Performance (TFlop/s)

Reduce memory footprint and data movement

Mixed precision LU factorization algorithm from Haidar et al:

- ▲ Dramatically more accurate compared to fp16 LU factorization
- ▼ Only marginally faster compared to half precision factorization algorithm
  $\Rightarrow$ poor exploitation of the tensor cores
- ▼ No reduction in memory footprint compared to full precision algorithm

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Idea: Store A in precision u_{16} and perform panel factorization and tensor core updates in precision u_{16} (TC16) in Haidar's algorithm

State-of-the-art mixed precision LU factorization

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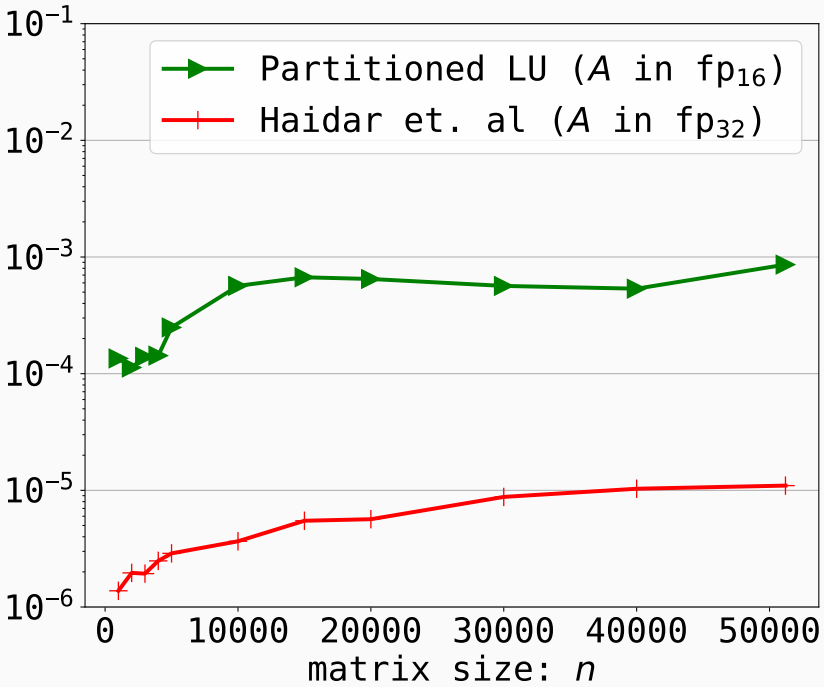

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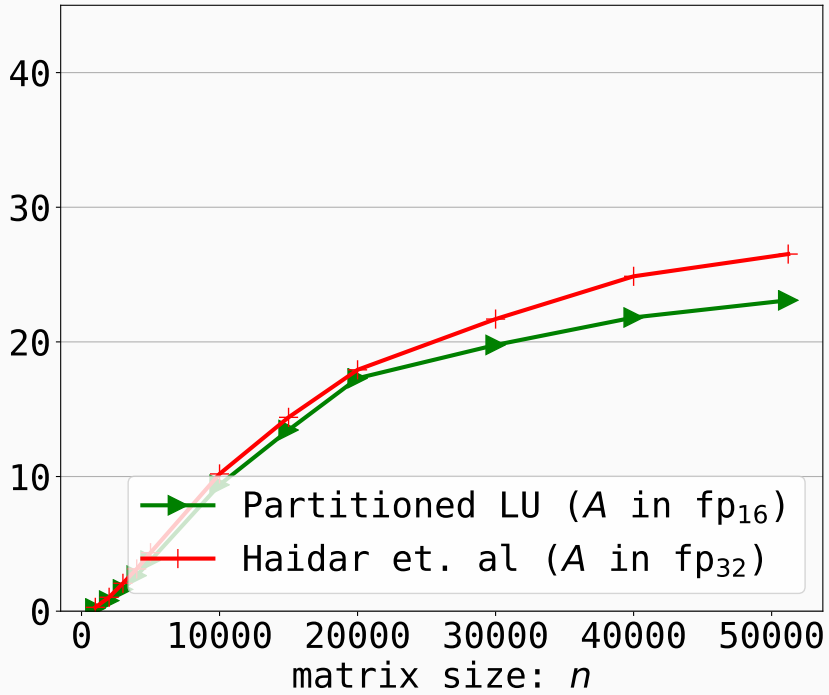

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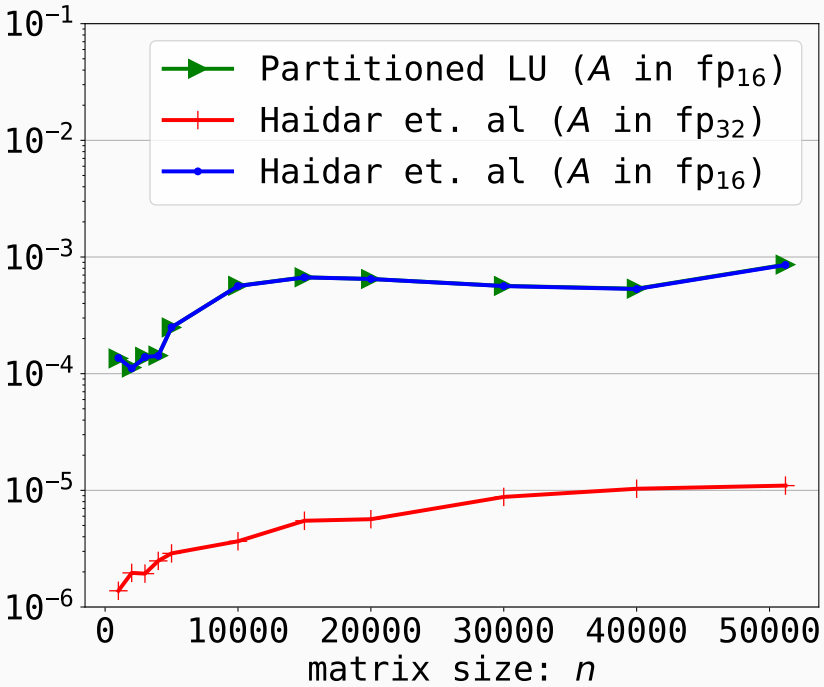
nu_{16}	$2u_{16} + nu_{32}$	$(n/4)u_{16}$



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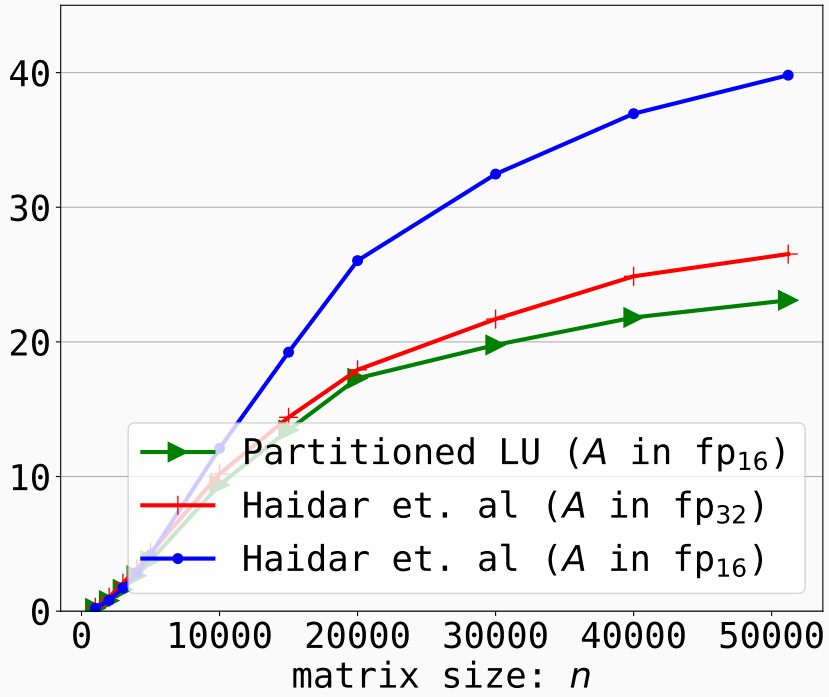

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with

$$\begin{aligned} B_{ij} &= \mathrm{fl}_{32}(A_{ij}) \\ B_{ij} &\leftarrow B_{ij} - \widetilde{L}_{ik}\widetilde{U}_{kj} \\ A_{ij} &\leftarrow \mathrm{fl}_{16}(B_{ij}) \end{aligned}$$

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Problem: In Haidar's algorithm, we need a f32 buffer corresponding to the trailing sub-matrix which, in the first few steps of the factorization is almost as large as the matrix itself.

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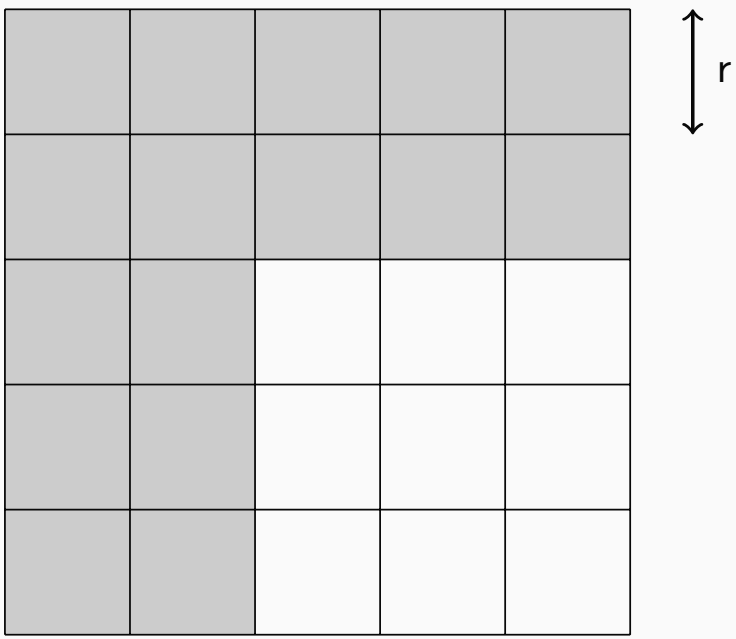
Problem: In Haidar's algorithm, we need a f32 buffer corresponding to the trailing sub-matrix which, in the first few steps of the factorization is almost as large as the matrix itself.

$\Rightarrow$ Switch to a left-looking factorization.

Partitioned LU factorization: left-looking variant

Left-looking LU factorization: at every step, the current panel is updated w.r.t the already computed factors (to the left) before being factored.



for $k = 1 : n/r$ **do**
  for $i = k : n/r$ **do**
    for $j = 1 : k - 1$ **do**
      Update $A_{ik} \leftarrow A_{ik} - L_{ij} U_{jk}$.
      Update $A_{ki} \leftarrow A_{ki} - L_{kj} U_{ji}$.
    end for
  end for
  Factorize $L_{kk} U_{kk} = A_{kk}$.
  for $i = k + 1 : n/r$ **do**
    Solve $L_{ik} U_{kk} = A_{ik}$ for L_{ik}.
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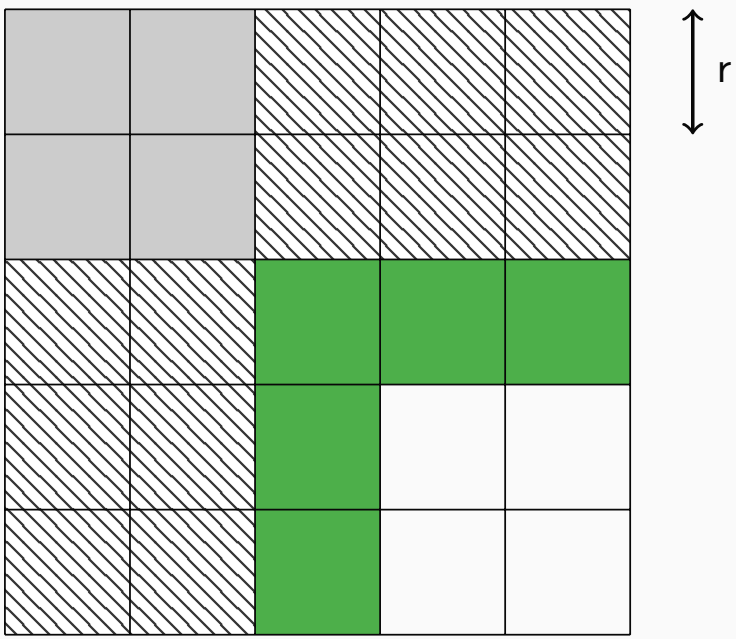

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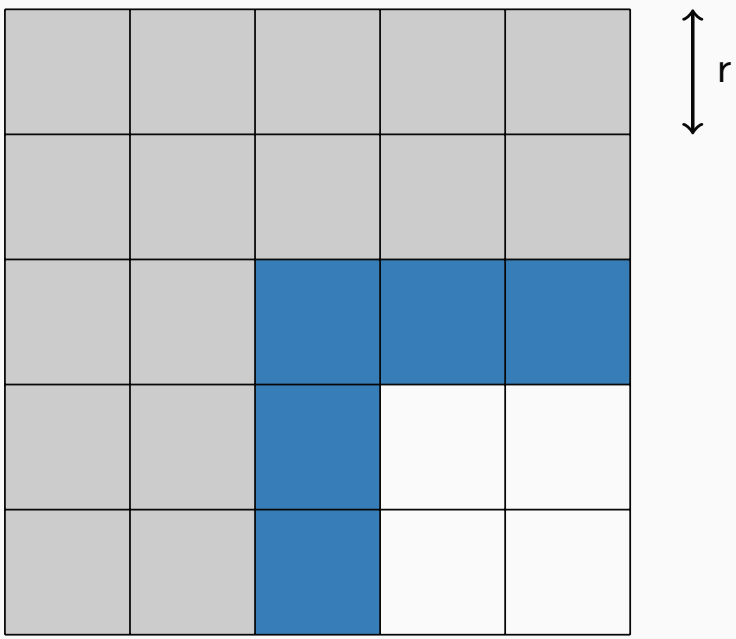

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  end for
end for

New left-looking mixed-precision LU factorization

A is now stored in precision u_{16}.

for $k = 1 : n/r$ **do**
  for $i = k : n/r$ **do**

    for $j = 1 : k-1$ **do**
      Update $A_{ik} \leftarrow A_{ik} - \widetilde{L}_{ij}\widetilde{U}_{jk}$
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  end for
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  for $i = k+1 : n/r$ **do**
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  end for

end for

New left-looking mixed-precision LU factorization

A is now stored in precision u_{16}. Using tensor cores $\Rightarrow$ TC16.

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    for $j = 1 : k-1$ **do**
      Update $A_{ik} \leftarrow A_{ik} - \widetilde{L}_{ij}\widetilde{U}_{jk}$ using tensor cores.
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  end for

end for

New left-looking mixed-precision LU factorization

A is now stored in precision u_{16}. Using tensor cores + fp32 buffer $\Rightarrow$ TC32

for $k = 1 : n/r$ **do**
  for $i = k : n/r$ **do**
    Convert to fp32: $B_{ik} = \mathrm{fl}_{32}(A_{ik})$ and $B_{ki} \leftarrow \mathrm{fl}_{32}(A_{ki})$.
    for $j = 1 : k-1$ **do**
      Update $B_{ik} \leftarrow B_{ik} - \widetilde{L}_{ij}\widetilde{U}_{jk}$ using tensor cores.
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  end for
  Factorize $L_{kk}U_{kk} = B_{kk}$ (in precision u_{32}).
  for $i = k+1 : n/r$ **do**
    Solve $L_{ik}U_{kk} = B_{ik}$ for L_{ik} (in precision u_{32}).
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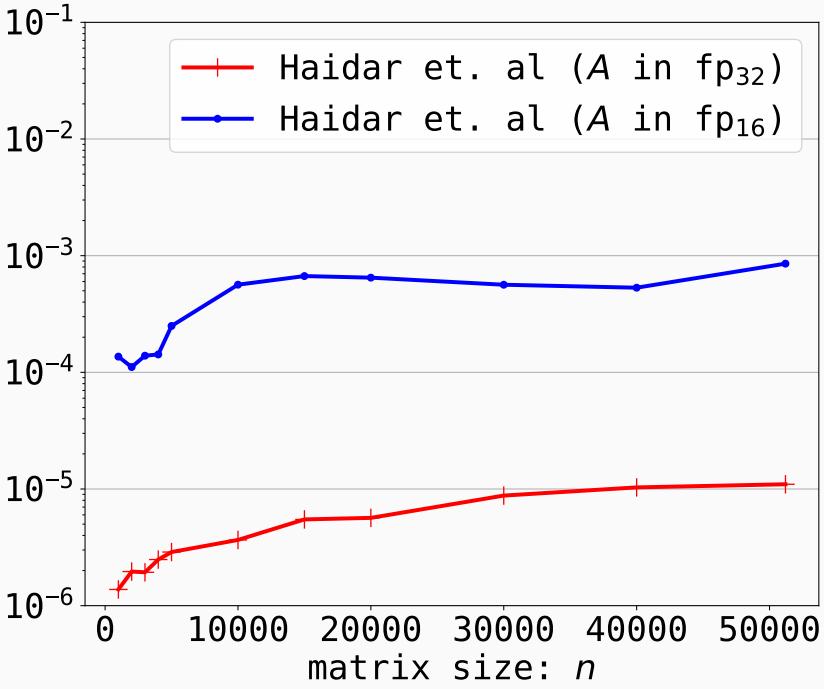
  end for
  for $i = k : n/r$ **do**
    Convert $\widetilde{L}_{ik} \leftarrow \mathrm{fl}_{16}(L_{ik})$ and $\widetilde{U}_{ki} \leftarrow \mathrm{fl}_{16}(U_{ki})$.
  end for
end for

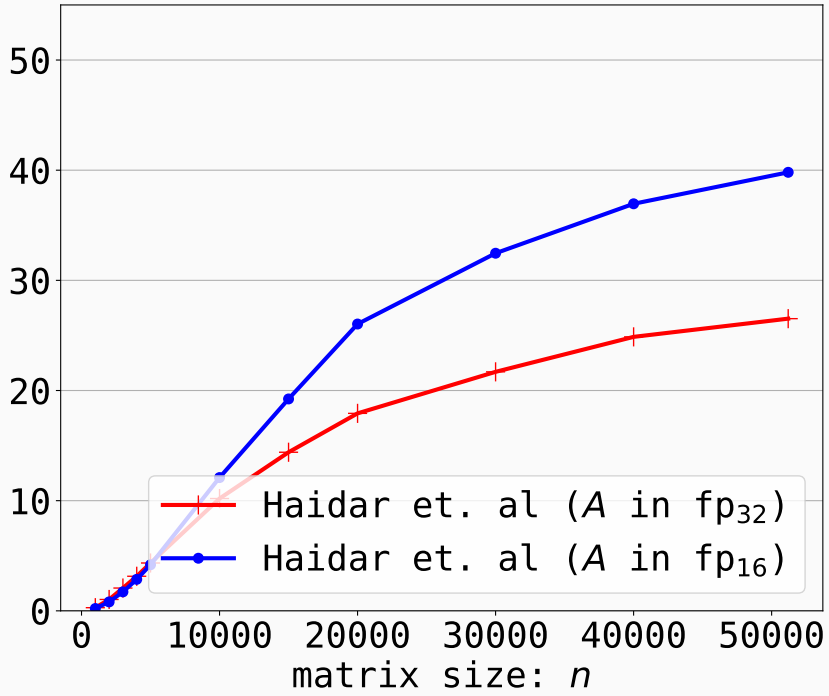
New left-looking mixed-precision LU factorization

Backward error bounds

Haidar et. al $u = u_{32}$	Haidar et. al $u = u_{16}$	New left-looking $u = u_{16}$
$2u_{16} + nu_{32}$	$(n/4)u_{16}$	



Backward error



Performance (TFlop/s)

New left-looking mixed-precision LU factorization

Backward error bounds

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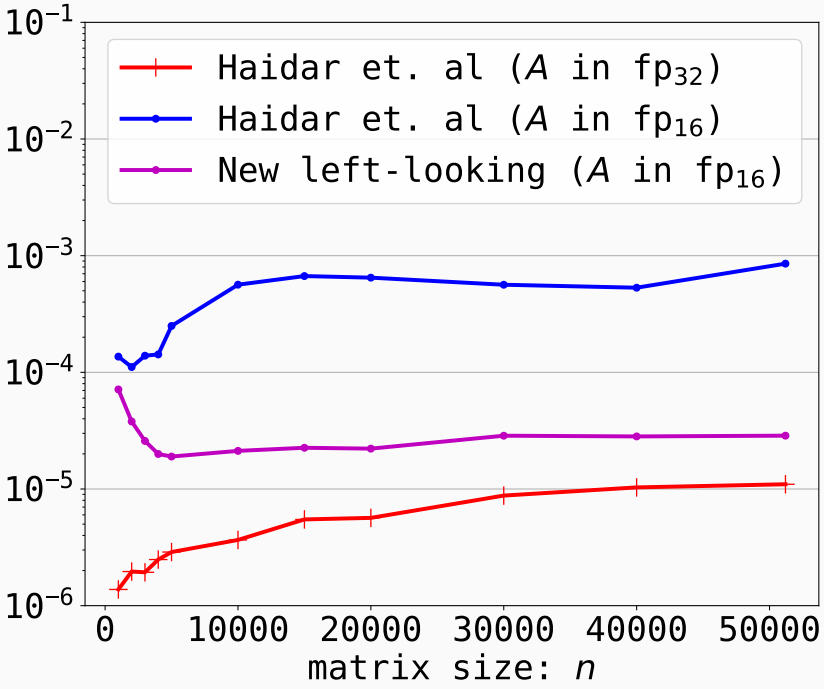

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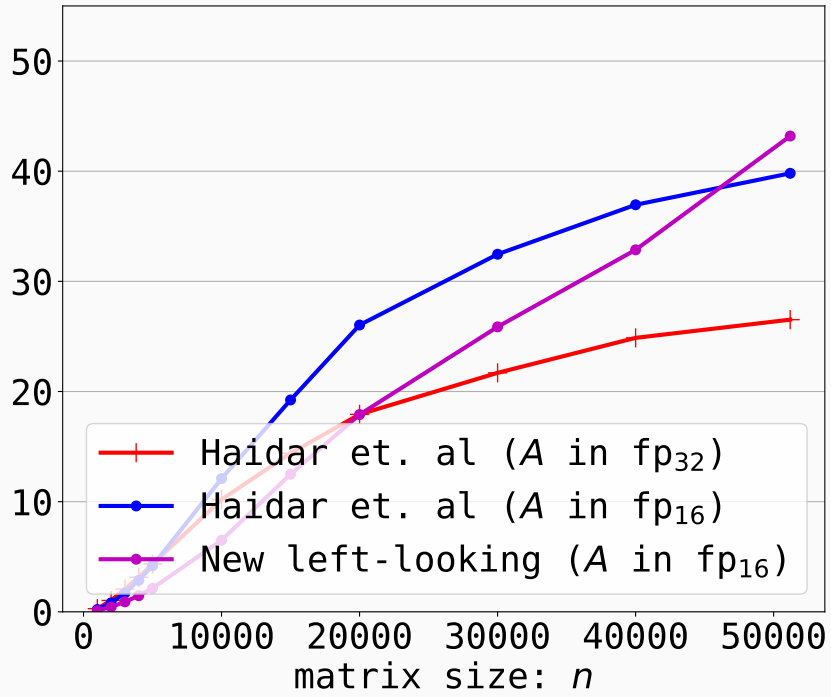

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fp32 storage (# of entries), r = 256

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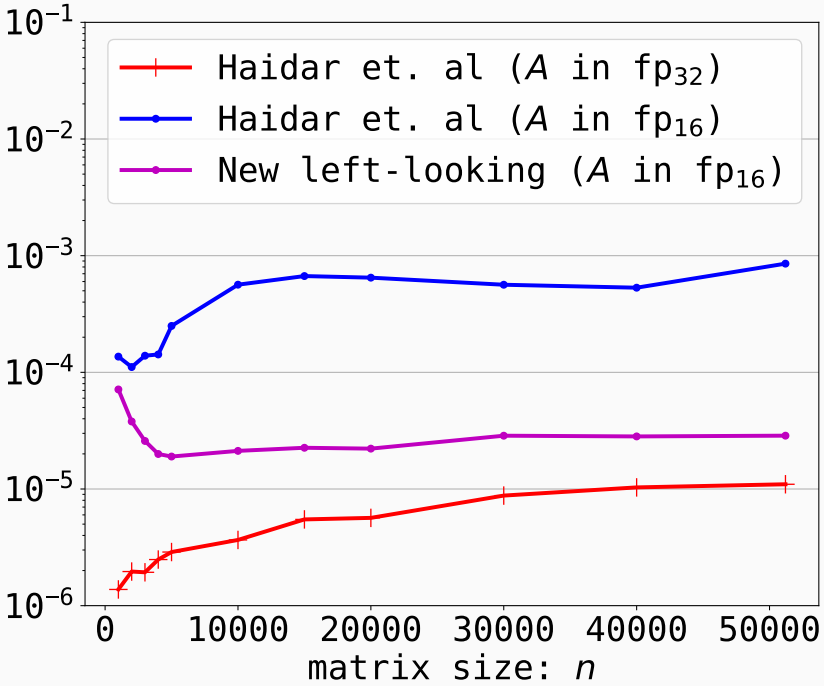
n^2	0	nr



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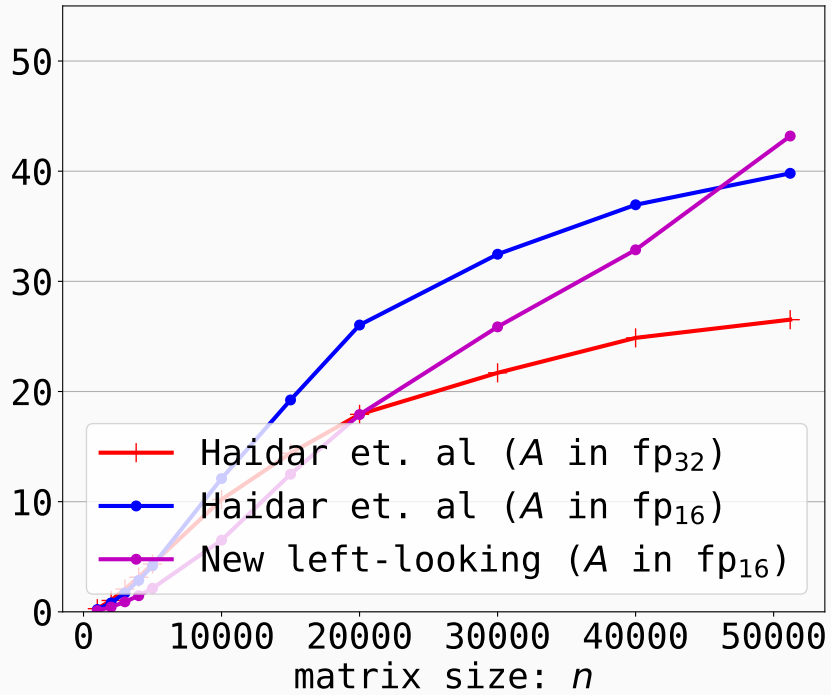

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New left-looking mixed precision LU factorization algorithm:

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  With $r = 256$, we have $nu_{32} \geq ru_{16}$ for $n \gtrsim 2.1 \times 10^6 \Rightarrow$ in practice, especially on single GPU kernel, ru_{16} dominates the error.

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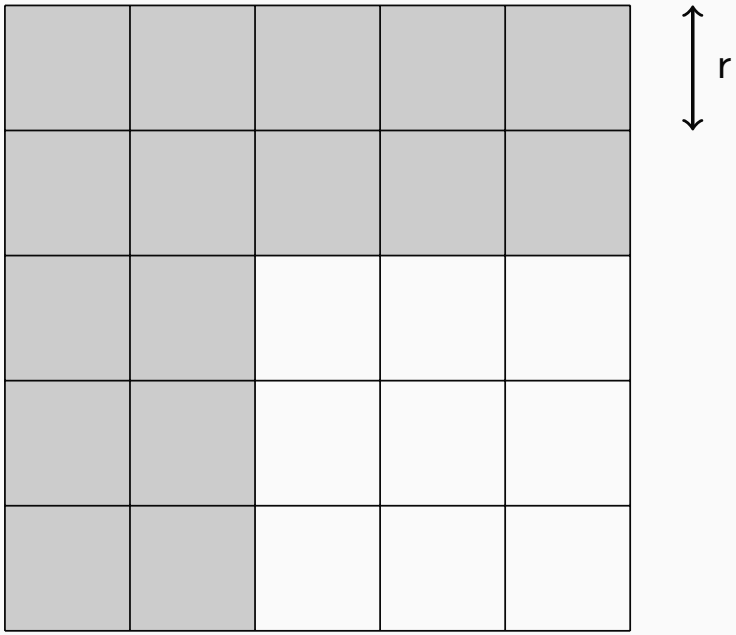
Idea:

- Partition the panels into a smaller blocks of size s.
- Accumulate the inner update into the fp32 buffer (left-looking) using the tensor cores (TC32).
- Perform the inner panel factorization in precision u_{32}.

New doubly partitioned left-looking algorithm



for $k = 1 : n/r$ **do**
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    Convert to fp32:
    $B_{ik} \leftarrow \mathrm{fl}_{32}(A_{ik})$ and
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      Using tensor cores (TC32):
      Update $B_{ik} \leftarrow B_{ik} - L_{ij} U_{jk}$.
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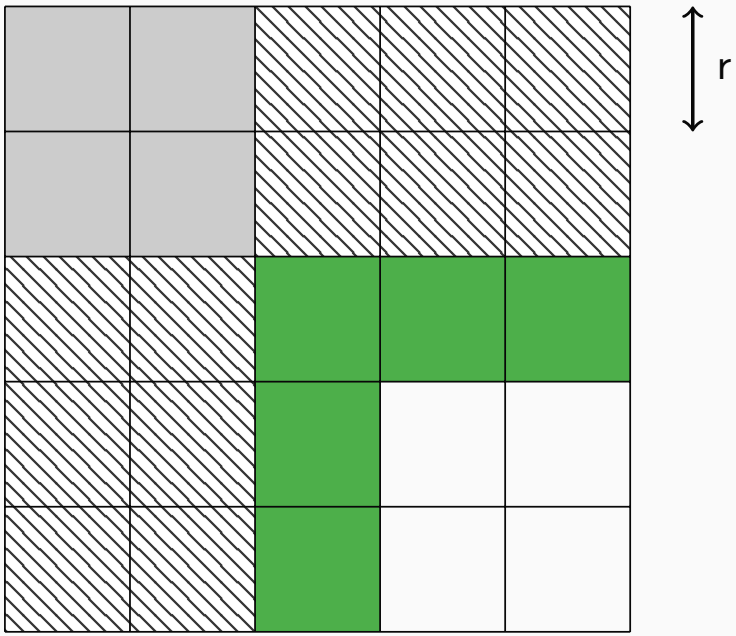

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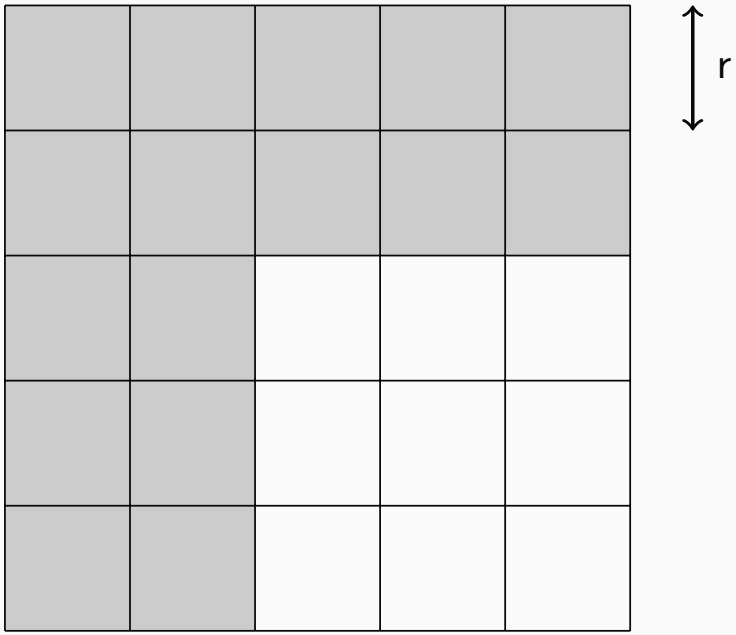

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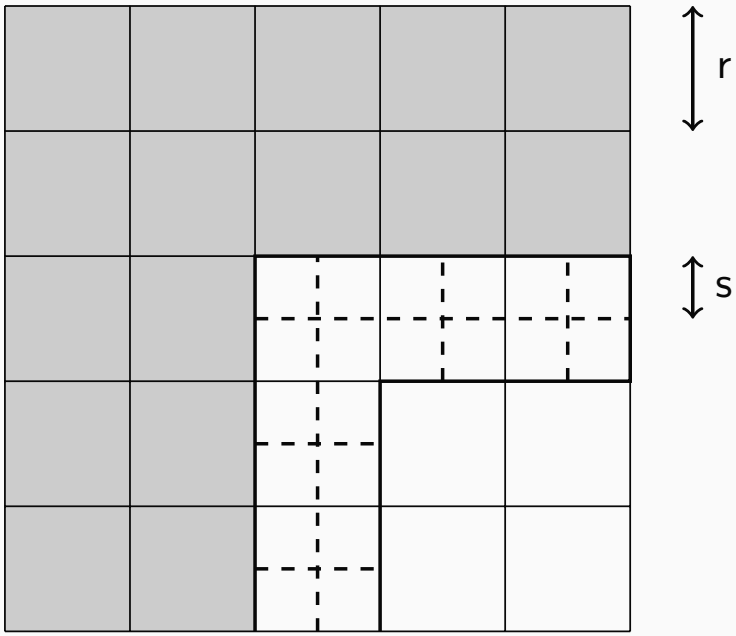

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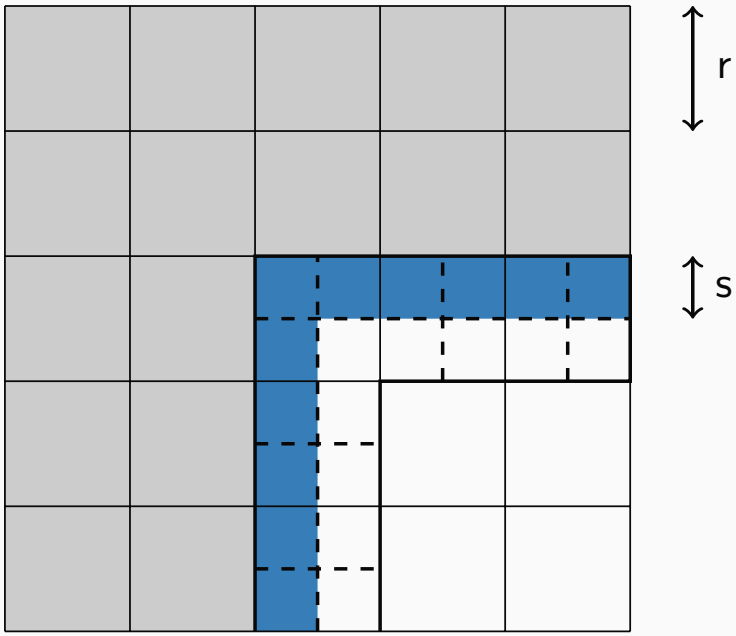

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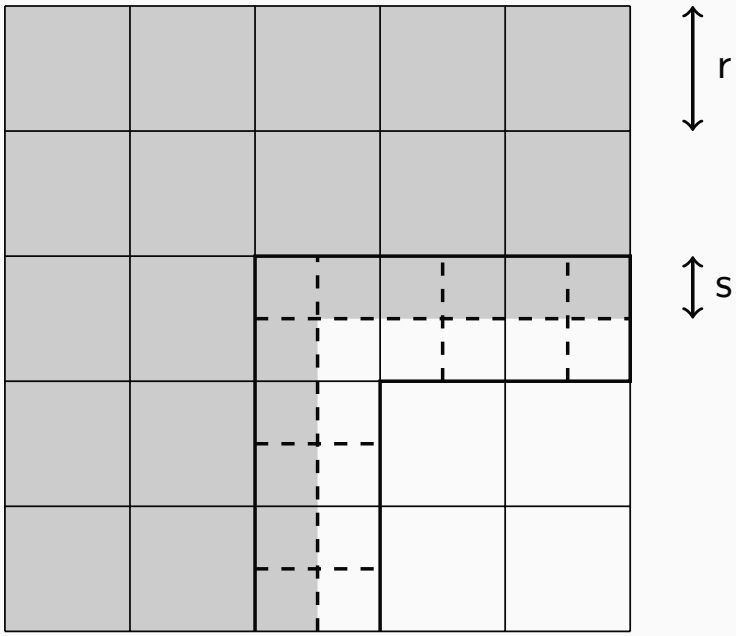

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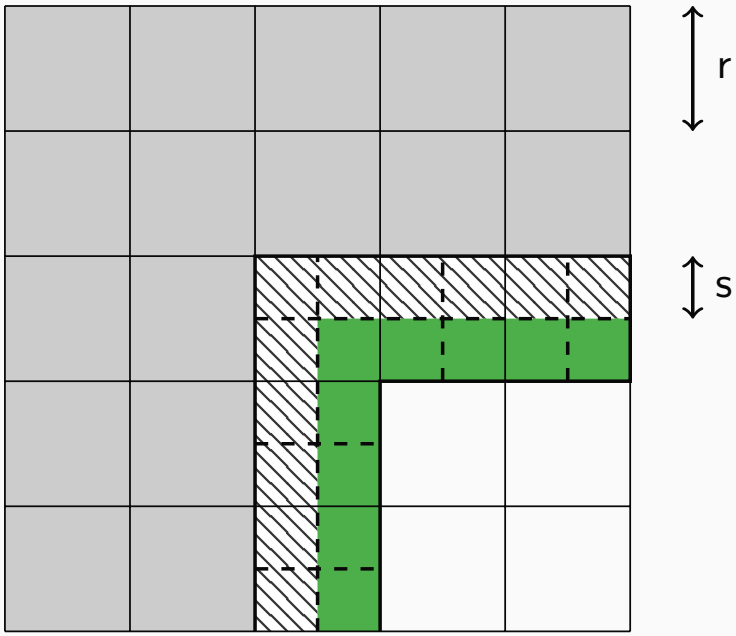

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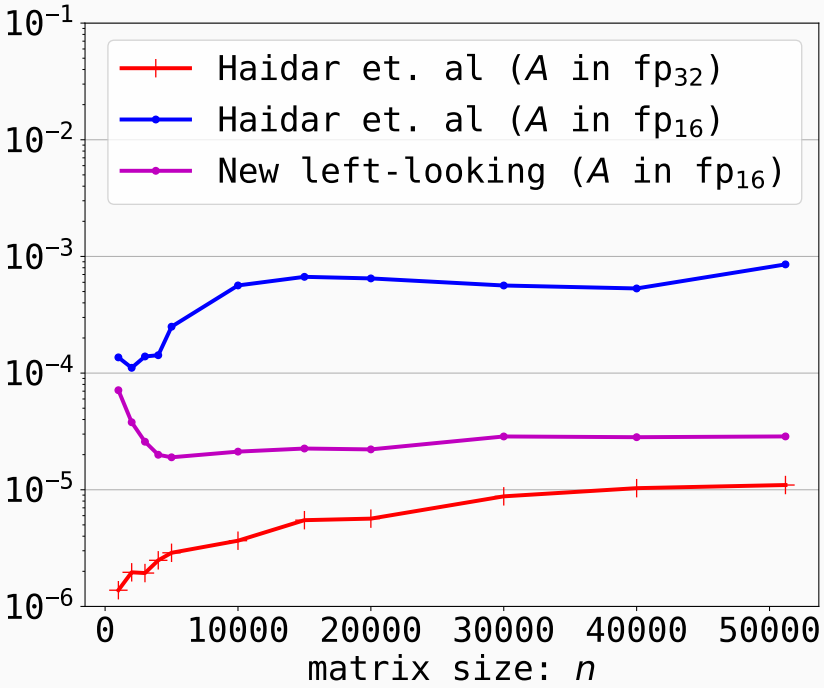

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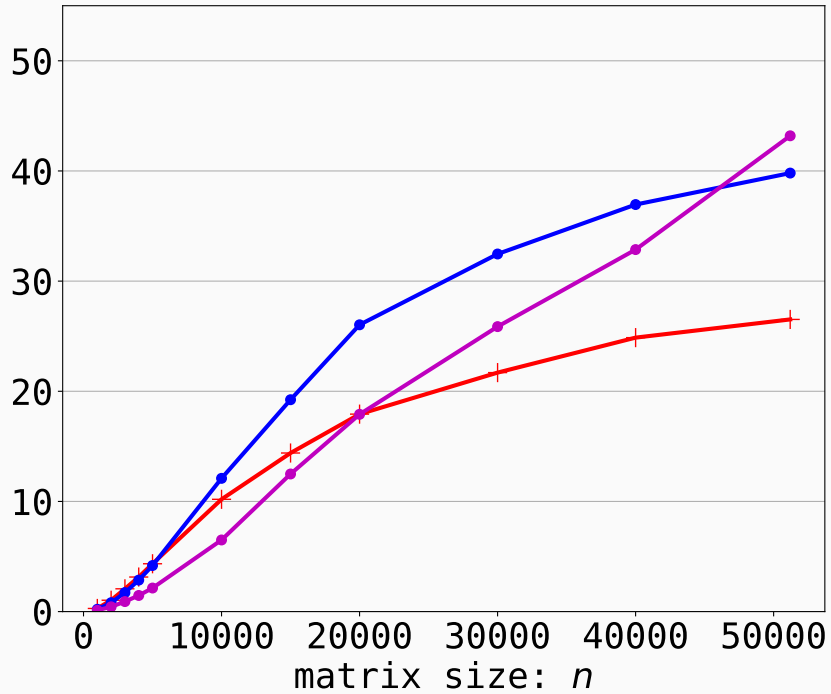

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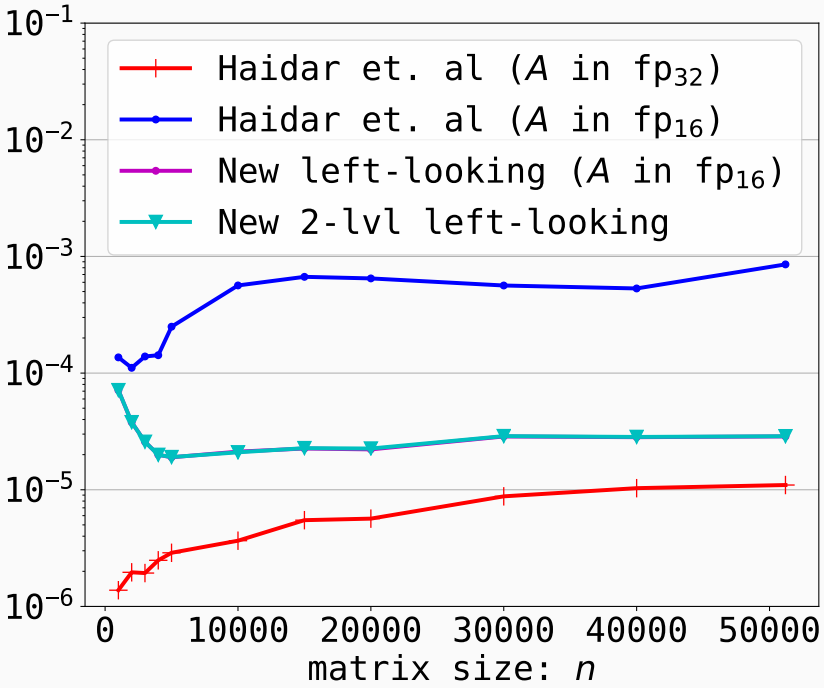

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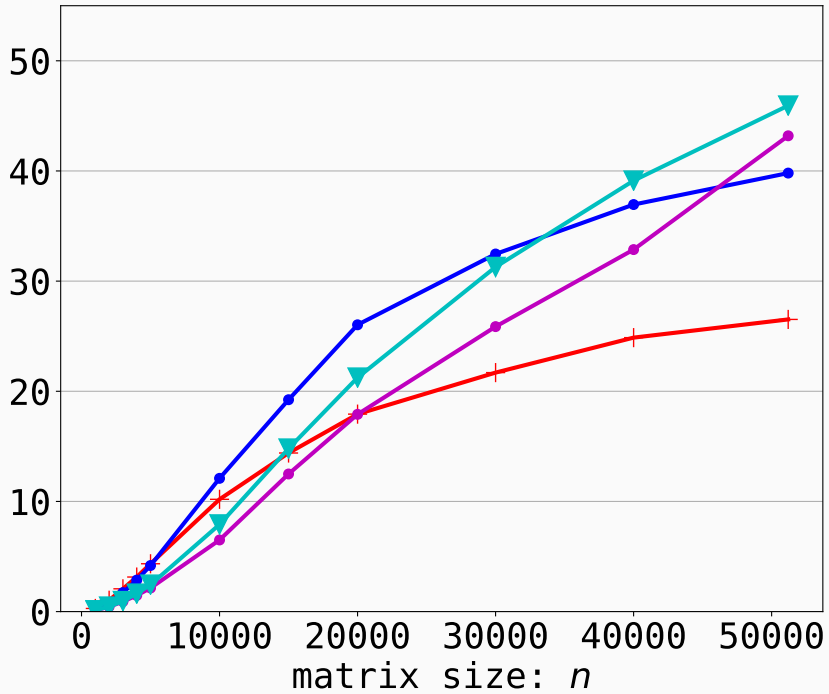

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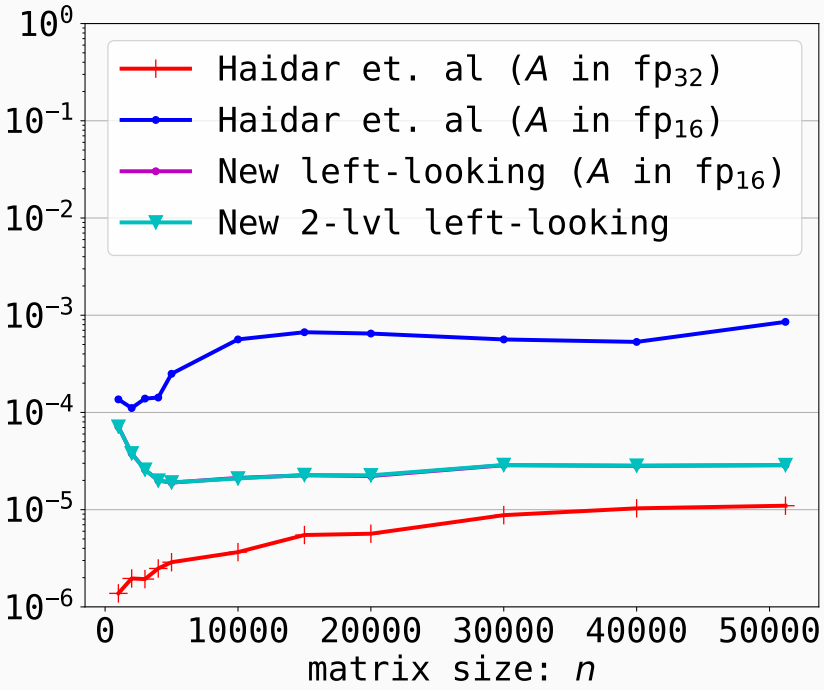

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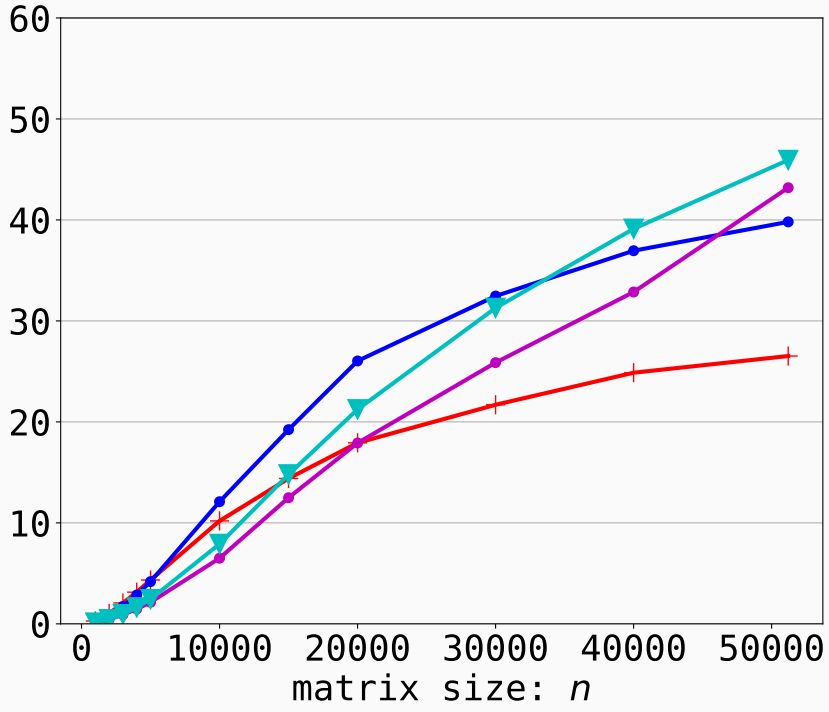

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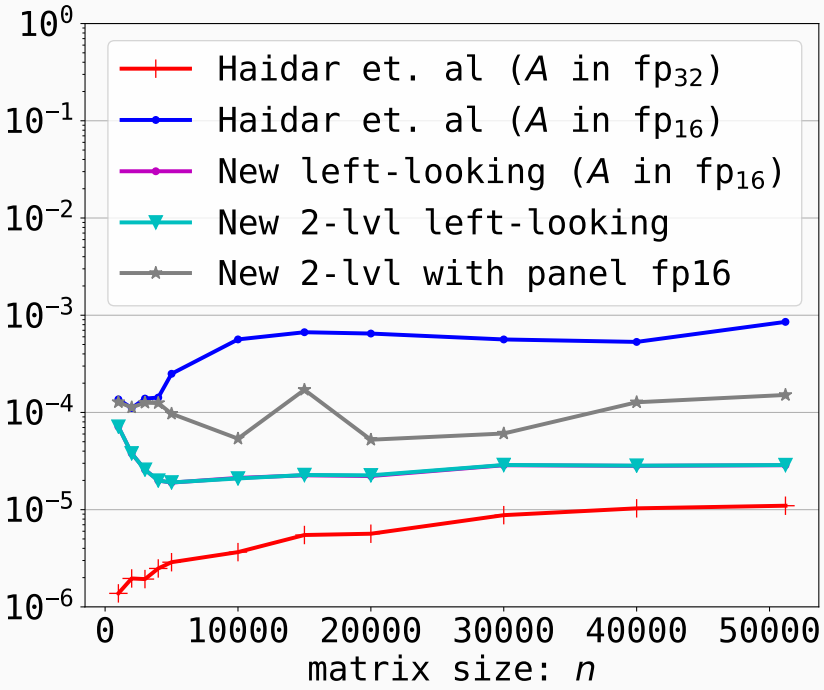

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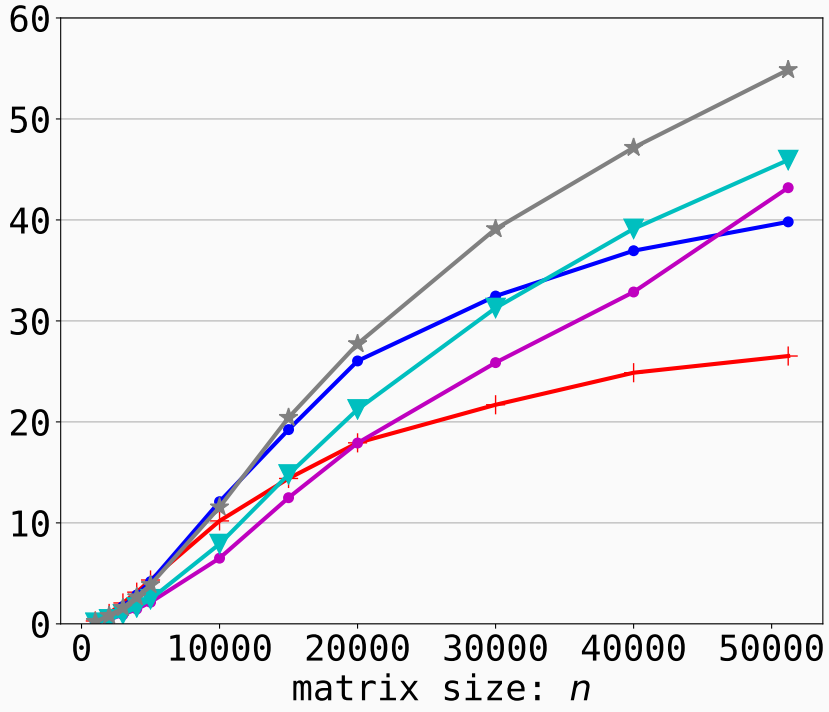

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Conclusions and future work

New mixed precision LU factorization algorithm:

- A stored in fp16 $\Rightarrow$ half the memory footprint.
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- Up to 2$\times$ faster (55 TFlop/s).
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See our preprint: F. Lopez and T. Mary, *Mixed Precision LU Factorization on GPU Tensor Cores: Reducing Data Movement and Memory Footprint* (MIMS EPrint 2020.20)

Thanks for listening!

Questions?